Quadratics

Quadratic equations:

A quadratic equation is a second-degree polynomial equation of the form ax^2 + bx + c = 0, where x represents an unknown variable, and a, b, and c are constants with a not equal to 0.

$ax^2 + bx + c$

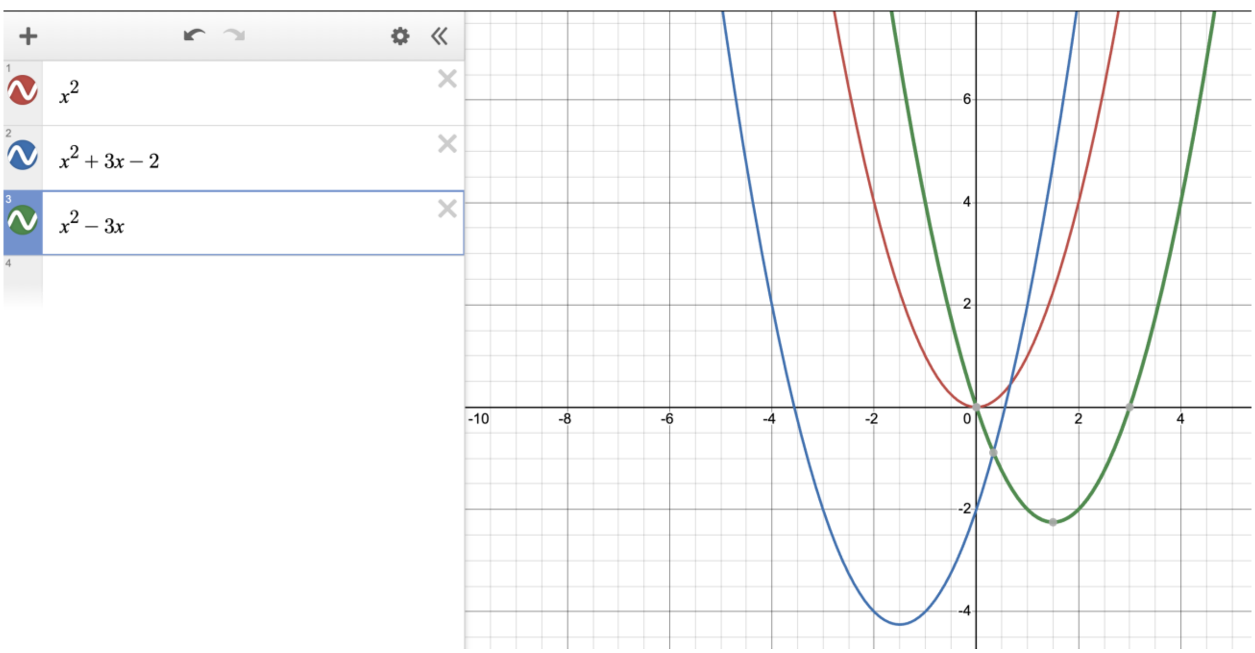
$x^2 + 4x + 2 \rightarrow a = 1, b = 4, c = 2$

$3x^2 - 2 \rightarrow a = 3, b = 0, c = -2$

$4x^2 - 5x + 32 \rightarrow a = 4, b = -5, c = 32$

$nx^2 + 5x - 2 \rightarrow a = n, b = 5, c = -2$

The graph different graphs:



Solving quadratic equations:

Factorization:

When solving quadratic equations there are many different techniques you can choose from. The first technique that will be discussed is the factorization method. When solving quadratics, you should always get 2 values of x because the graph is curved in a way where 2 values of x can have the same y value. The way you solve by factorization is you find two values which multiply to equal the c value in the equation and add up to equal the b value in the equation. Here is a basic example:

$$x^2 + 2x - 8$$

$$factors\ of\ 8 \rightarrow 1,2,4,8$$

$$the\ two\ values\ which\ add\ up\ to\ 2\ are\ 4,2 \rightarrow 4 - 2 = 2$$

$$then\ you\ place\ them\ into\ the\ brackets\ respectively$$

$$(x+4)(x-2) \rightarrow to\ check\ the\ factorisation\ is\ \mathrm{corr}\,ect\ \exp\ and\ this$$

$$Equate\ each\ bracket\ to\ 0 \rightarrow x + 4 = 0,\ x = -4,\ and\ x - 2 = 0,\ x = 2$$

In this example you have to find the factors of –8 which would add up to equal 2. Hence the values you can use are 4, and –2. Then you can place them into the brackets as shown; to double check you can re-expand the brackets and see if you get the same original equation. Then to finally solve the equation you equate each bracket to 0 respectively to find the values of x. Here is another example:

$x^2 - 7x + 10 = 0$
$factors\ of\ 10 = 1, 2, 5, 10$
$-5 - 2 = -7$
$(x-5)(x-2) = 0$
$(x-5) = 0,\ and\ (x-2) = 0$
$x = 5,\ and\ x = 2$

Now when the value of a in the equation is more than one there is a different case but it is similar. You will multiply a to c and find factors of the value which add up to b in the equation. After finding that you will split the middle term of b then normally factorize the right and left side of the equation. Then the first bracket is the common bracket from both sides and the remainder values will go into the second bracket. You can then equate both bracket to 0 to solve for x. Here are some examples:

$2x^2 + 7x - 15 = 0$
$2(-15) = -30$
$factors\ of\ -30\ that\ add\ up\ to\ 7 = 10,\ and\ -3$
$2x^2 + 10x - 3x - 15$
$2x^2 + 10x \vert -3x - 15 \vert$
$2x(x+5) + (-3(x+5)) = (2x-3)(x+5) \rightarrow x = \frac{3}{2},\ and\ x = -5$

In this example, I multiplied a to c to get –30, then I found the factors of –30 which add up to the b value which is 7 -> that was 10 and –3. Then I replaced 7x with 10x-3x which would simplify to 7x anyways. Then to find the factorization I split the equation in half and factored both sides respectively. I found that x+5 was common, so I took it as one of the brackets and the remainder 2x-3 in the other brackets. Now that I had the factorization, I equated both brackets to 0 to find the two values of x which were –5 and 3/2.

$$6x^2 - 10x - 4 = 0$$

$$6(-4) = -24, factors\ of\ -24\ that\ add\ up\ to\ -10 = -12 + 2$$

$$6x^2 - 12x + 2x - 4$$

$$6x^2 - 12x| + 2x - 4|$$

$$6x(x-2) + (2(x-2)) = (6x+2)(x-2)$$

$$6x + 2 = 0 \rightarrow x = -\frac{2}{6}, and\ x - 2 = 0 \rightarrow x = 2$$

<u>Completing the square method:</u>

Completing the square method is another way to solve quadratic equations, it is often used when creating quadratic graphs. The way you do this is take c to the other side of the equation. Then you divide the b value by 2 and square it. This new value will be added to the equation on the left side and the original b value will stay the same. Then you subtract the same halved and squared value you found on the right side of the equation. Then you can normally factorize the left side of the equation, then you solve for x or bring the c value from the right to the left again to make it simpler. Here are some examples:

$$x^2 - 4x + 8 = 0$$

$$\left(x^2 - 4x\right) = -8 \rightarrow \frac{-4}{2} = -2, -2^2 = 4 \rightarrow \left(x^2 - 4x + 4\right) - 4 = -8$$

$$\left(x^2 - 4x + 4\right) = (x-2)^2 = -12 \rightarrow (x-2)^2 + 4$$

$$(x-2)^2 = -4 \rightarrow x - 2 = \sqrt{-4} \rightarrow x = 2 \pm i\sqrt{4}$$

-4/2=-2

A trick to this factorization is that the halved value of b will always be the value added to the bracket getting squared.

$$x = 2 \pm i\sqrt{4}$$

In this example, the discriminant value (the value in the square root) is negative meaning that x has no real roots.

$$x^2 + 6x + 14 = 0$$

$$\left(x^2 + 6x\right) = -14 \rightarrow \frac{6}{2} = 3, 3^2 = 9 \rightarrow \left(x^2 + 6x + 9\right) - 9 = -14$$

$$(x+3)^2 = -5 \rightarrow x + 3 = \sqrt{-5} \rightarrow x = -3 \pm i\sqrt{5}$$

$$x = -3 \pm i\sqrt{5}$$

<u>The quadratic formula:</u>

The quadratic formula is very easy to operate with, it is simply a formula to remember which you can add to a calculator to find the values of x. Here is the quadratic formula:

$$Quadratic\ Formula = \frac{\left(-b \pm \sqrt{b^2 - 4(a)(c)}\right)}{2a}$$

Here are some examples of using the quadratic formula:

$$x^2 + 3x - 10 = 0$$

$$b = 3, a = 1, c = -10$$

$$\frac{\left(-3 \pm \sqrt{3^2 - 4(1)(-10)}\right)}{2} \rightarrow \frac{\left(-3 \pm \sqrt{49}\right)}{2}$$

$$x = \frac{(-3+7)}{2} = 2, or\ x = \frac{(-3-7)}{2} = -5$$

$$2x^2 + 6x - 20 = 0$$

$$a = 2, b = 6, c = -20$$

$$\frac{\left(-6 \pm \sqrt{6^2 - 4(2)(-20)}\right)}{4} \rightarrow \frac{\left(-6 \pm \sqrt{196}\right)}{4}$$

$$x = \frac{(-6+14)}{4} = 2, and\ x = \frac{(-6-14)}{4} = -5$$

Difference of two squares:

When the quadratic equation has no b value it is very simple to solve for x. However, if you want to factorize it you must find the square root of the c value which will be the value added to the brackets of the factorized version of the equation. There will be two brackets with a ± value of the square root. Here are some examples:

$x^2 - 4 = 0$

$x^2 = 4$

$x = \sqrt{4} = \pm 2 \rightarrow x = \pm 2$

In this equation you can automatically tell that there will be no middle term (b value) because the ±2 will cancel out.

$to\ Factorize: \sqrt{4} = \pm 2$

$\left(x+2\right)\left(x-2\right) = x^2 + \underline{\left(2x - 2x\right)} - 4 \rightarrow x^2 - 4$

$x^2 - 36$

$x^2 = 36 \rightarrow x = \sqrt{36} = \pm 6 \rightarrow x = \pm 6$

$to\ Factorize: \sqrt{x^2} = x, \sqrt{36} = \pm 6$

$\left(x+6\right)\left(x-6\right) = x^2 + 6x - 6x - 36 \rightarrow x^2 - 36$

$$4x^2 - 49$$

$$4x^2 = 49 \rightarrow \frac{49}{4} = 12.25 \rightarrow x^2 = 12.25 \rightarrow x = \sqrt{12.25} \rightarrow x = \pm 3.5$$

$$to\ Factorize: \sqrt{4x^2} = 2x,\ and\ \sqrt{49} = \pm 7$$

$$(2x - 7)(2x + 7) = 4x^2 + 14x - 14x - 49 \rightarrow 4x^2 - 49$$

Deriving quadratic equations from word problems and solving them:

Often in exams you get context to a scenario, and you are asked to form an expression which satisfies the conditions of the context. This way you need to understand how to derive quadratic equations from word problems. Here are some solved examples:

Example 1:

"Emily is planning to build a rectangular garden in her backyard. She wants the length of the garden to be 3 meters longer than its width. If the area of the garden is 24 square meters, find the dimensions of the garden."

Here is what we can denote from the problem:

$$\text{width} = x,\ \text{length} = x + 3,\ area = 24m^2,\ area = l \cdot w$$

$$if\ area = l \cdot w,\ 24 = (x+3)(x) \rightarrow 24 = x^2 + 3x$$

$$turn\ it\ into\ a\ proper\ quadratic\ equation\ to\ solve{:}\ x^2 + 3x - 24 = 0$$

You can use any quadratic method, I will use the quadratic formula:

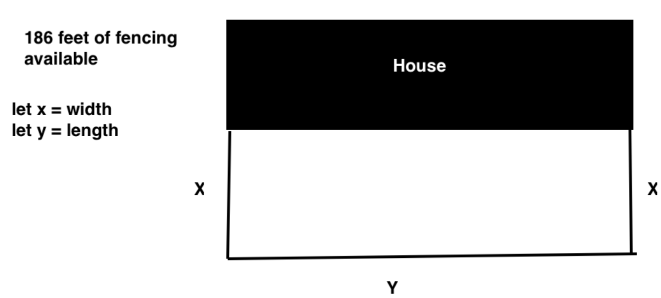
$$\frac{\left(-3 \pm \sqrt{3^2 - 4(1)(-24)}\right)}{2} = x = 3.62m,\ or\ x = -6.62m$$

Since the length of anything cannot be a negative value in reality, the only applicable value for x is 3.62m, so the dimensions of the garden are the following: width = 3.62m, and length = 6.62m

<u>Example 2:</u>

“Chang decides to fence his backyard, he has 186 feet of fencing, and one side of the backyard is facing the wall of his house so there are only 3 sides to consider. Considering the maximum amount of fencing what is the maximum possible area of the backyard?

Diagram of backyard:



Now that we know how the backyard looks, we can denote an equation to work with:

$the\ perimeter\ of\ the\ backyard: 2x + y = 186$

$area\ of\ backyard: (xy) = A$

$rearrange\ perimeter\ \exp ression: y = -2x + 186$

$sub\ it\ into\ area\ \exp ression: x(-2x + 186) = A$

$-2x^2 + 186x = A$

$Since\ we\ are\ solving\ \text{for}\ the\ \max imum\ dimensions\ we\ use\ vertex\ point$

$vertex = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right) \rightarrow -\frac{186}{2(-2)} \rightarrow x = 46.5ft$

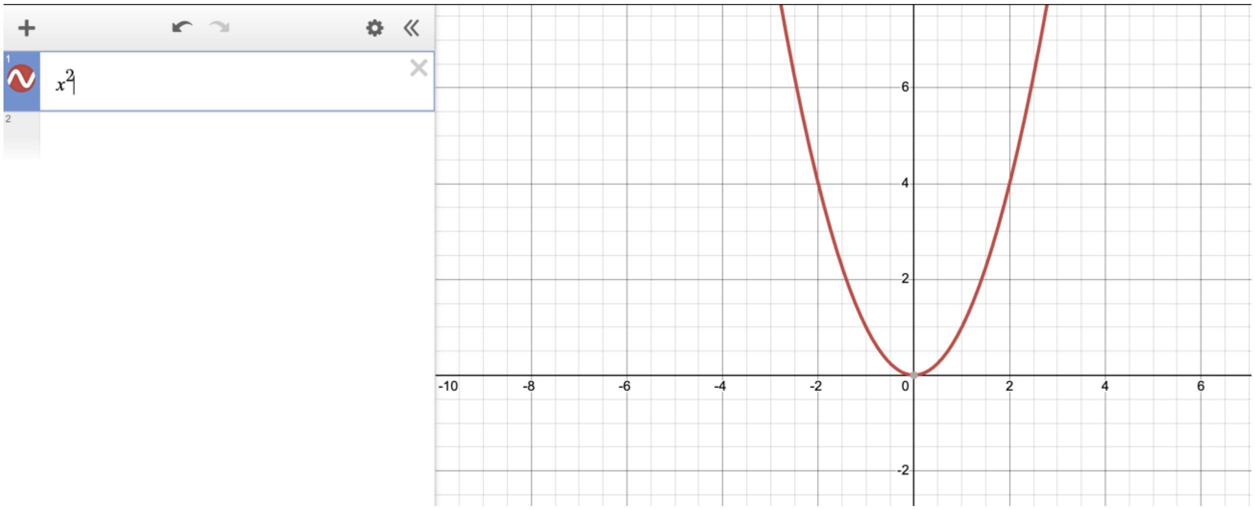
$x = 46.5ft\ so\ sub\ it\ in\ to\ find\ y: 2(46.5) + y = 186 \rightarrow y = 93ft$

$multiply\ them\ together\ to\ find\ the\ \max\ area: 93(46.5) = 4324.5ft^2$

Quadratic graphs

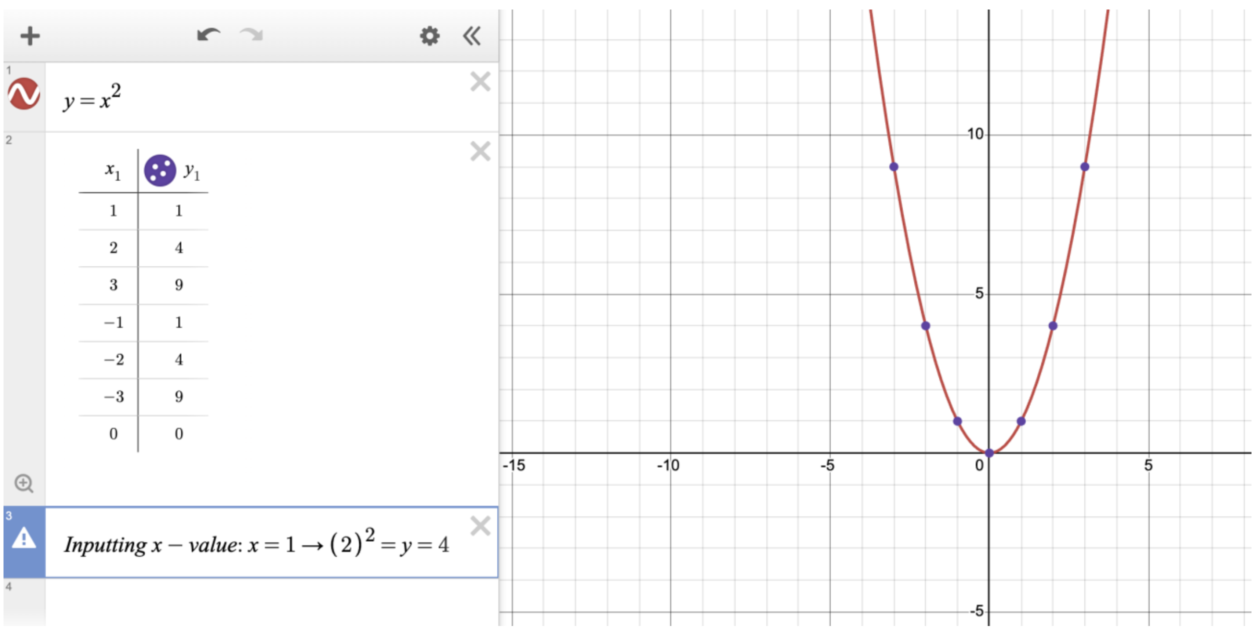
How to create quadratic graphs:

There are many ways to create quadratic graphs. It is known that quadratic graphs always take a similar shape to the letter U. You can make a table for x and y to find the corresponding outputs for each input, plot the points and join them to create a graph. Or you can use the vertex formula.



X and Y table method:

You can create a table of x and y, take random values for x, input them into the equation to find the value of y then plot these points on a graph. You can then join these different points to create the curve. Here is an example:



Vertex equation method:

When given an equation, you can convert the equation into a vertex form to find the coordinates of the minimum/maximum point of the graph, then you can factorize or solve the equation to find the root of x, and finally the y-intercept. When you have these 4 points you can create the graph of the equation. Here is an example:

$x^2 - 4x + 3$

$Vertex\ Form\ is\!: (x - k)^2 + h$

$Now\ use\ completing\ square\ method\ to\ find\ the\ vertex\ Form\!:$

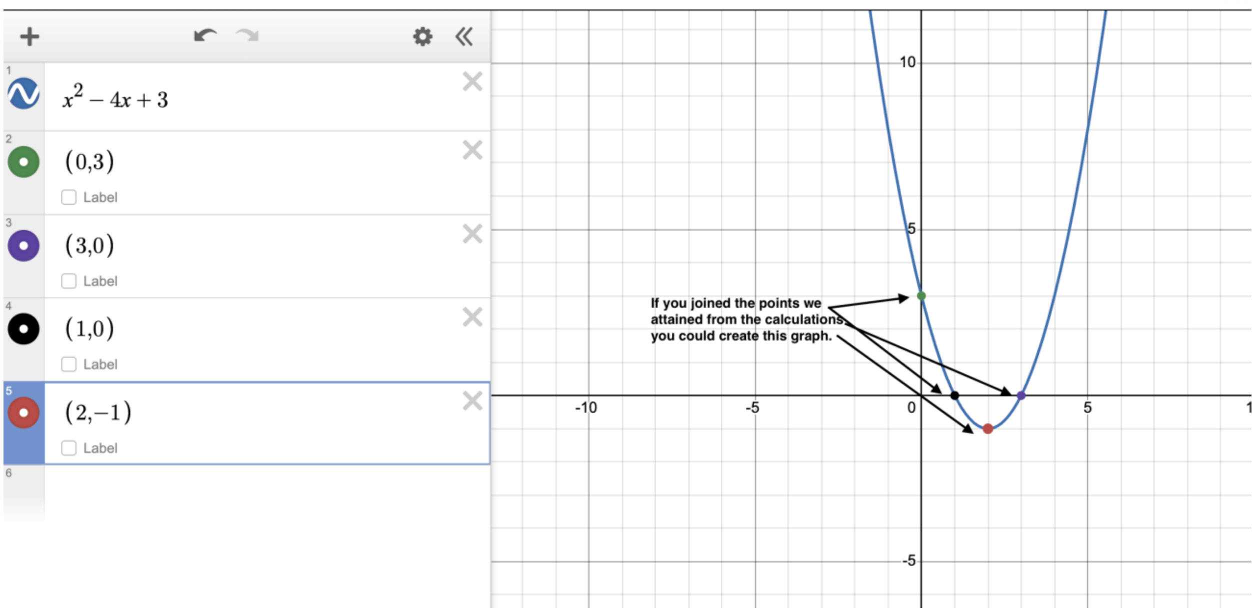
$\left(x^2 - 4x + 4\right) - 4 = -3 \to (x - 2)^2 - 4 + 3 = 0 \to (x - 2)^2 - 1$

$The\ vertex\ is\ always\ (k, h) \to (2, -1)$

$Now\ find\ the\ roots\ of\ x, I\ will\ use\ factorization\!:$

$(x - 3)(x - 1) \to x - 3 = 0 \to x = 3,\ and\ x - 1 = 0 \to x = 1$

$The\ y\ intercept\ by\ subbing\ 0\ in\ x\!: y = (0)^2 - 4(0) + 3 \to y = 3 \to (0{,}3)$



Simultaneous equations

Simultaneous equations:

Simultaneous equations, also known as a system of equations, refer to a set of two or more equations that share the same variables. The goal when solving simultaneous equations is to find the values of the variables that satisfy all of the equations in the system. These equations can be solved using various methods such as substitution, elimination, or matrix methods.

Example:

"John buys 2 bananas and 5 apples for a total of $10, Mary buys 5 bananas and 2 apples worth $12.4. Work out the value of a single banana and a single apple."

In this example, we are presented with information related to items being bought in a store. In this case they are specifically looking into bananas and apples. Since the bananas and apples would have the same value in the store for both customers you can create two equations with the two same variables representing bananas and apples to solve for the price of each item as asked in the question.

$$2\ bananas + 5\ apples\ =\ \$10 \rightarrow 2b + 5a\ =\ 10$$

$$5\ bananas + 2\ apples\ =\ \$12.4 \rightarrow 5b + 2a\ =\ 12.4$$

$$Then\ you\ would\ solve\ \text{ for }\ each\ Variable$$

Since we do not know the rules of solving this system of equations, so we will come back to this question. First, let's understand the different methods of solving these equations.

Basic elimination method:

In the basic elimination method, you will have two systems of equations where two variables are present. If the same variable from both equations has the same value, you can subtract or add the two equations to create one basic equation to solve by eliminating the common variable. This way you can solve for the second variable and come back to the eliminated variable and solve it as well. Here are some examples:

$3x + 5y = 21$

$3x + 2y = 12$

$3x \text{ is common on both sides } \rightarrow 3x - 3x = 0$

$(3x + 5y = 21) - (3x + 2y = 12) = (3y = 9)$

$3y = 9 \rightarrow y = 3$

$3x + 5(3) = 21 \rightarrow x = 2 \rightarrow x = 2, \text{ and } y = 3$

$5x + 2y = 24$

$2x - 2y = 4$

$2y \text{ is common} \rightarrow 2y + (-2y) = 0$

$(5x + 2y = 24) + (2x - 2y = 4) = (7x = 28)$

$7x = 28 \rightarrow x = 4 \rightarrow 5(4) + 2y = 24 \rightarrow 2y = 4 \rightarrow y = 2$

$x = 4, \text{ and } y = 2$

Complex elimination method:

Sometimes, the system of equations has common variables but none of them are common. So, to apply the elimination method, you can multiply either one or both equations with a factor which will produce common variables. After attaining those common variables, you can apply the basic elimination method anyways. Here are some examples:

$3x + 4y = 13.5$

$2x - 6y = -10.5$

$\text{Multiply } 3x \text{ and } 2x \rightarrow 3x(2) = 6x, \text{ and } 2x(3) = 6x$

$2(3x + 4y = 13.5) - (3(2x - 6y = -10.5)) = (26y = 58.5)$

$26y = 58.5 \rightarrow y = 2.25$

$\text{sub it in} \rightarrow 3x + 4(2.25) = 13.5 \rightarrow x = 1.5, \text{ and } y = 2.25$

$3n - 4c = 22$

$5n + 2c = 54$

$4c(2) = 8c, \text{ and } 2c(4) = 8c \rightarrow 2(3n - 4c = 22) + 4(5c + 2c = 54)$

$(6n - 8c = 44) + (20c + 8c = 216) = (26n = 260)$

$n = \frac{260}{26} = 10 \rightarrow \text{sub it in} \rightarrow 3(10) - 4c = 22 \rightarrow n = 2$

$c = 2, \text{ and } n = 10$

Substitution method:

The substitution method is commonly used when equations get harder like when you have a system of equations with quadratics. However, they can be used in basic linear equations as well. All you do is change one variable to make it the subject of the equation then you can substitute the expression into the second equation and solve for the common variable. Here are some examples:

$$x^2 + y = 18$$

$$y + x = 6$$

$$\text{Make } y \text{ the subject to keep things simple} \rightarrow y = -x + 6$$

$$\text{Sub it in} \rightarrow x^2 + (-x + 6) = 18 \rightarrow x^2 - x + 6 = 18$$

$$\text{Solve for } x \rightarrow x^2 - x - 12 \rightarrow (x - 4)(x + 3) \rightarrow x = 4, \text{ or } -3$$

$$\text{Sub both values in to find two possible solutions of } y \rightarrow (4)^2 + y = 18$$

$$y = 2, \text{ and } (-3)^2 + y = 18 \rightarrow y = 9$$

When it comes to quadratic system of equations there can be two possible values for both variables as shown here unless in the context of the equations given it is not feasible to have a negative or positive value for a variable. In that case you only consider one of the values and its corresponding outcome with the second unknown variable.

$$x^3 - y = 25$$

$$x + y = 5$$

$$Make\ y\ the\ subject \rightarrow y = -x + 5 \rightarrow sub\ it\ in \rightarrow x^3 + x - 5 = 25$$

$$solve \rightarrow x^3 + x - 30 = 0 \rightarrow x^3 + x = 30 \rightarrow (x-3)\left(x^2 + 3x + 10\right)$$

$$x - 3 = 0 \rightarrow x = 3,\ and\ \frac{\left(-3 \pm \sqrt{3^2 - 4(10)}\ \right)}{2} \rightarrow no\ real\ roots$$

$$This\ \text{mean}\,s\ that\ this\ system\ of\ equations\ only\ have\ one\ real\ value\ \text{for}\ x$$

$$Now\ sub\ it\ in \rightarrow (3)^3 - y = 25 \rightarrow y = 2$$

$$x = 3,\ and\ y = 2$$

In this example, I had to factorize the cubic equation, then solve for both brackets to find the values of x. The quadratic equation produced imaginary number values so there was only one real root of x which was 3.

Complex system of equations:

Sometimes, if the problem gets harder you can get a system of equations with more than two variables, then you can use both elimination and substitution or just one of them to solve the problem. Your objective is normally to eliminate one variable then proceed to solve the two equations as they have two variables using the substitution method or elimination method again. Here are some examples:

$$x - 3y + 3z = -4$$

$$2x + 3y - z = 15$$

$$4x - 3y - z = 19$$

$$Eli\min ate\ one\ Variable\ First: (x - 3y + 3z = -4) + (2x + 3y - z = 15)$$

$$\rightarrow 3x + 2z = 11 \rightarrow now\ next\ equation:$$

$$(2x + 3y - z = 15) + (4x - 3y - z = 19) = 6x - 2z = 34$$

$$Now\ use\ the\ two\ equations: (3x + 2z = 11) + (6x - 2z = 34)$$

$$9x = 45 \rightarrow x = 5$$

$$Now\ sub\ it\ in \rightarrow 3(5) + 2z = 11 \rightarrow z = -2$$

$$Now\ sub\ both\ values\ in\ to\ find\ y: 2(5) + 3y - (-2) = 15 \rightarrow y = 1$$

$$x = 5, z = -2, and\ y = 1$$

<u>How to change the subject of an equation</u>

<u>Changing the subject of an equation:</u>

When asked to change the subject, you are basically rearranging the equation to find out the expression which is equivalent to a specific variable you wish to use.

You can do so by applying the basic concept of BODMAS. When given an equation you can move variables and values around with addition and subtraction. If it is more complicated, you can always use division and multiplication as well as factorization to help you single out a variable you wish to make the subject. Here are some examples:

$$\textit{Make v the subject of the equation}:$$

$$3 + v - z = 5y$$

$$\textit{Subtract 3 from both sides} \rightarrow v - z = 5y - 3$$

$$\textit{Then add z to both sides}: v = 5y - 3 + z$$

$$\textit{Make n the subject of the equation}:$$

$$5n + 2y = 2n - 4$$

$$\textit{Take all n values to one side}: 5n - 2n = -4 - 2y \rightarrow 3n = -4 - 2y$$

$$\textit{Divide both sides by 3 to find n}: n = \frac{(-4 - 2y)}{3}$$

Now some more complex examples:

$Make\ x\ the\ subject\ of\ the\ equation:$

$3\sqrt{x+b} = a + b$

$Divide\ both\ sides\ by\ 3: \sqrt{x+b} = \frac{(a+b)}{3}$

$Square\ both\ sides: x + b = \left(\frac{(a+b)}{3}\right)2 \rightarrow x + b = \left(\frac{(a+b)^2}{9}\right)$

$Subtract\ b\ from\ both\ sides: x = \left(\frac{a^2 + 2ab + b^2}{9}\right) - b$

$make\ b\ the\ subject:$

$(3+b)^2 - y = \sqrt{n} + 3$

$Take\ y\ to\ the\ other\ side: (3+b)^2 = \sqrt{n} + 3 - y$

$find\ the\ square\ root\ on\ both\ sides: 3 + b = \sqrt[4]{n} \pm \sqrt{3-y}$

$take\ 3\ to\ the\ other\ side: b = -3 \pm \sqrt[4]{n} \pm \sqrt{3-y}$

$you\ can\ simplify\ the\ right\ side\ Further: b^2 = 9 \pm \sqrt{n} + 3 - y$

$b^2 = 12 - y \pm \sqrt{n} \rightarrow b = \sqrt{12-y} \pm \sqrt[4]{n}$